Optimum Growth

The problem is to maximize the value function of a society with infinite time horizon, that is to maximize the integral of discounted utility by choosing a path of optimum consumption decision for every instant of time. The dynamic system is described as

$$\begin{aligned} \max_{\{c(t)\}_{t=0}^{\infty}} \quad & \mathcal{W} = \int_0^\infty u(c)e^{-rt}dt \\ \text{u.d.N.} \quad & \dot{k} = f(k) - c - k(\delta + n) \\ & k_0 = k(0) \\ & 0 \leq c(t) \leq f[k(t)] \end{aligned} \tag{1}$$

The present value hamiltonian is defined as

$$\mathcal{H}(t, k, c, \lambda) := u(c)e^{-rt} + \lambda[f(k) - c - (n + \delta)k]. \tag{2}$$

With the current value multpiplier defined as $\lambda := \mu e^{-rt}$, (2) becomes

$$\mathcal{H}(t, k, c, \mu) = e^{-rt}\{u(c) + \mu[f(k) - c - (n + \delta)k]\}. \tag{3}$$

The first order conditions are provided by Pontryagin's Maximum Principle and the resulting law of motion for the state, co-state und the inner solution condition reads as

$$\dot{k} = \frac{\partial \mathcal{H}}{\partial \mu}e^{rt} = f(k) - c - k(\delta + n) \tag{4}$$

$$\dot{\mu} = -\frac{\partial \mathcal{H}}{\partial k}e^{rt} + r\mu = -\mu[f'(k^*) - (n + \delta + r)] \tag{5}$$

$$\frac{\partial \mathcal{H}}{\partial c} = u'(c^*) - \mu^* = 0 \Leftrightarrow \mu^* = u'(c^*) \tag{6}$$

The transversality conditions are

$$\lim_{t\to\infty} \mu(t)e^{-rt}k(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \mathcal{H}(t, \cdot) = 0. \tag{7}$$